Petri Nets with Catalysts

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This work was produced in part on
occupied Cahuilla and Tongva land.

Applied Category Theory
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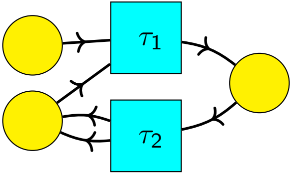
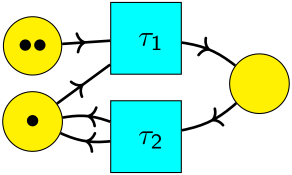
Petri Nets

- resources:  e.g. $\mathrm{H_2O}$
- processes:  e.g. a reaction



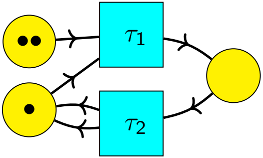
Markings

- resources: ◯ e.g. H_2O
- processes: □ e.g. a reaction
- tokens: • e.g. a molecule of H_2O



$$2A + 1B$$

Executions



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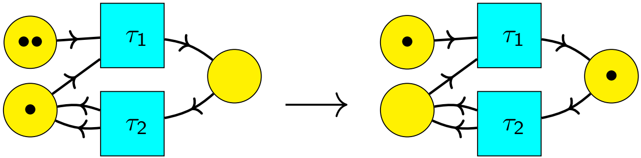

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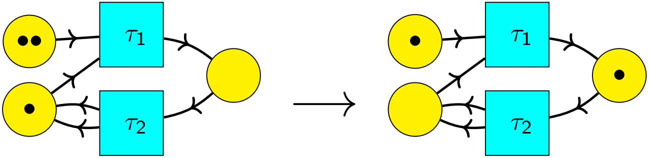
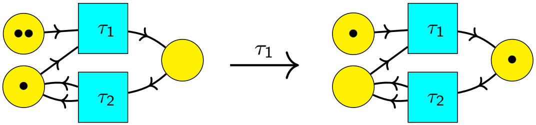
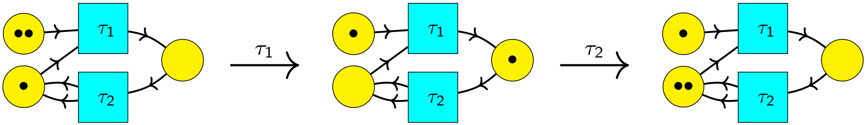


$$2A + 1B \xrightarrow{\tau_1} 1A + 1C$$

Sequential Execution



Sequential Execution



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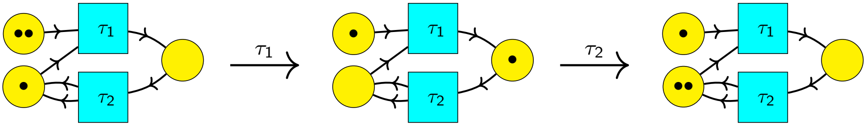


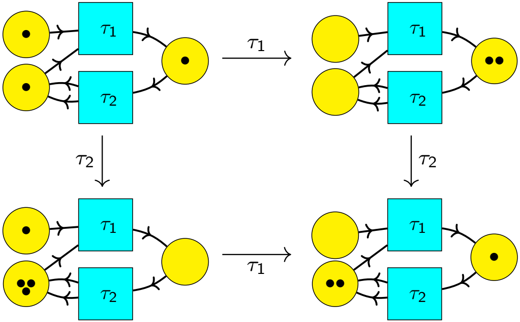
$$2A + 1B \xrightarrow{\tau_1} 1A + 1C \xrightarrow{\tau_2} 1A + 2B$$

Definition

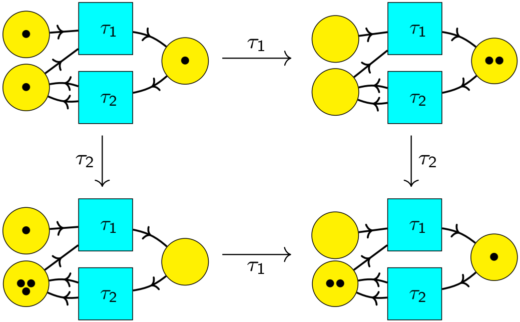
For a Petri net P, define a category FP where

- *an object is a marking*
- *a morphism is a sequence of executions*
- *composition is given by concatenation.*

Concurrent Execution

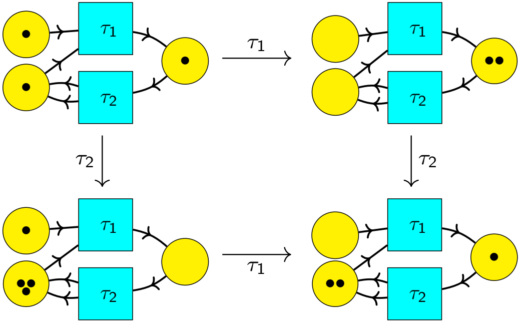


Concurrent Execution



$$1A + 1B + 1C \xrightarrow{\tau_1 + \tau_2} 2B + 1C$$

Concurrent Execution

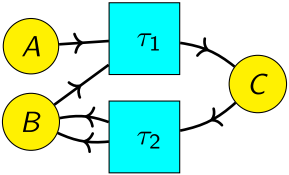


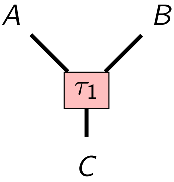
$$1A + 1B + 1C \xrightarrow{\tau_1 + \tau_2} 2B + 1C$$

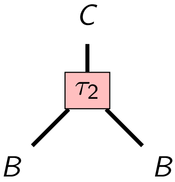
Definition

Give FP a (commutative) monoidal structure + as above.

String Diagrams

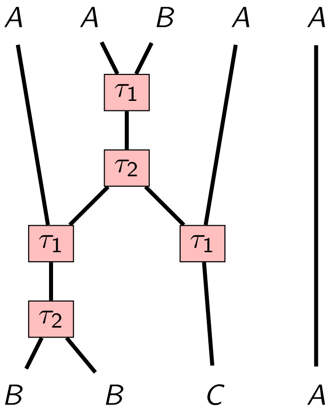






String Diagrams

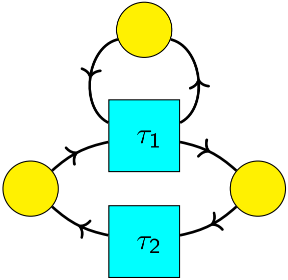
A morphism in *FP*:



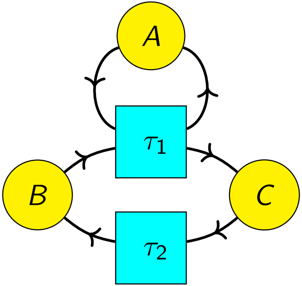
Catalysts

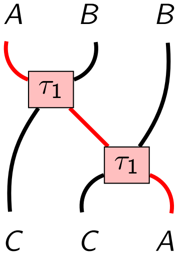
Definition

A **catalyst** *in a Petri net is a resource whose in-degree and out-degree relative to each transition are equal.*

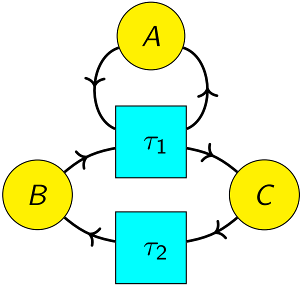


Catalysts





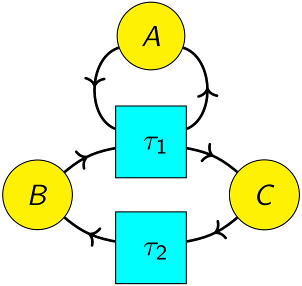
Catalysts



$A \quad B \quad B$

$C \quad A \quad A$

Catalysts



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Proposition (Baez, Foley, M)

For a Petri net P with catalysts, as categories we get

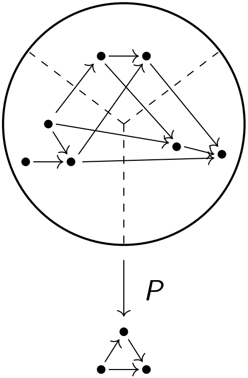
$$FP = \coprod_{c \in \mathrm{Catalysts}} FP_c .$$

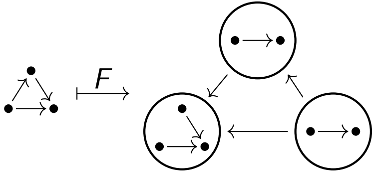
In particular, this gives a monoidal opfibration $FP \to \mathbb{N}[C]$.

Theorem

The Grothendieck construction gives a 2-equivalence:

$$\mathsf{Fib}(\mathcal{X}) \cong \mathsf{ICat}(\mathcal{X})$$

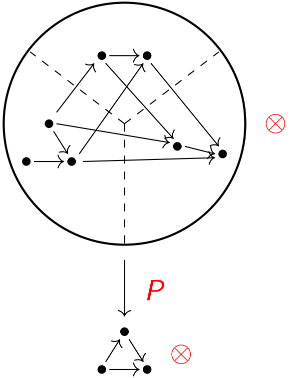


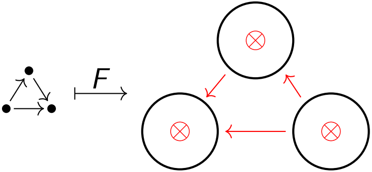


Monoidal Grothendieck Construction

Theorem (Shulman; Vasilakopoulou, M)

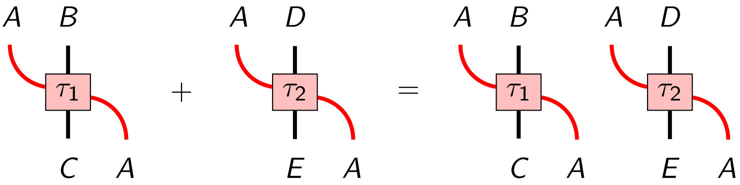
If X is (co)cartesian monoidal, then monoidal (op)fibrations over X are equivalent to X-(op)indexed monoidal categories.





Not monoidal subcategories

FP_{1A} is not closed under $+$:

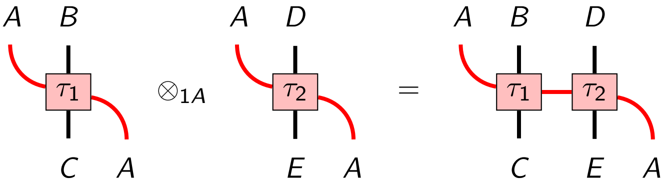


We have a monoidal opfibration
$$\begin{array}{c} FP \\ \downarrow \\ \mathbb{N}[C] \end{array}$$

but not a cocartesian base...

Structure on fibres

We should be reusing the catalyst! In FP_{1A}:



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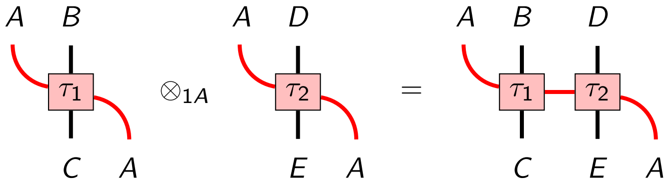

This is only PREmonoidal though

Network Models

- FP is monoidally opfibred over $\mathbb{N}[C]$: $$\begin{array}{c} FP \\ \downarrow \\ \mathbb{N}[C] \end{array}$$

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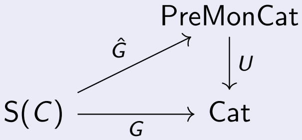
- Let $\mathsf{S}[C]$ denote the free symmetric monoidal category on C
$$\mathsf{S}[C] \to \mathbb{N}[C] \xrightarrow{p} \mathsf{Cat}$$

Theorem (Baez, Foley, M.)

The global monoidal indexed category $G\colon \mathsf{S}(C) \to \mathsf{Cat}$ *lifts to a functor* $\hat{G}\colon \mathsf{S}(C) \to \mathsf{PreMonCat}$*:*



Network Models

- monoidal functor

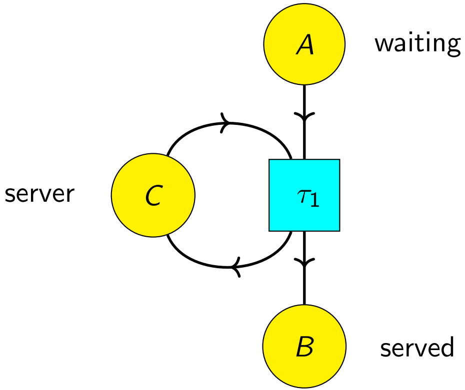
$$\mathsf{S}[C] \xrightarrow{\hat{G}} \mathsf{PreMonCat}$$

- monoidal Grothendieck construction gives a monoidal category
  - objects = same objects as FP, markings
  - morphisms = sequential executions + permutations of catalyst tokens
  - tensor = concurrent execution + permutation sum

  this gives a variant of the category FP which models **individual token philosophy** on the catalyst tokens, and **collective token philosophy** on all others

Future

- applications to queueing theory



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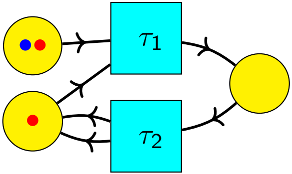

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- what about other fibrations of FP?

John C. Baez, John Foley, and Joe Moeller.
Network models from petri nets with catalysts.
Compositionality, 1(4), 2019.

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