The Grothendieck construction and structured categories

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Motivation

$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

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$$\mathsf{Mod}_{all}???$$

Motivation

Grothendieck: Yes!

- objects (k, M), where $M \in \mathrm{Mod}_k$
- maps $(f, g)\colon (k, M) \to (k', M')$ where $f\colon k \to k'$ and $g\colon M \to f^*(M')$

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$f \colon k \to k'$

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$$\mathsf{Mod}\colon \mathsf{Ring} \to \mathsf{Cat}???$$

$$f\colon k \to k' \rightsquigarrow f^*\colon \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

Motivation

$$k \in \mathsf{Ring} \rightsquigarrow \mathsf{Mod}_k \in \mathsf{Cat}$$

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

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Given

$$\mathsf{Mod}\colon \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$$

we defined Mod_{all} to have

- objects (k, M), where $M \in \mathsf{Mod}_k$
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Given

$$\mathcal{F}\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

we define $\int \mathcal{F}$ to have

- objects (x, a), where $x \in \mathcal{X}$, $a \in \mathcal{F}(x)$
- maps $(f, g)\colon (x, a) \to (x', a')$ where $f\colon x \to x'$ and $g\colon a \to \mathcal{F}f(a')$

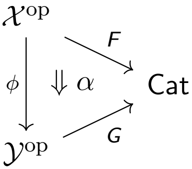
Indexed Categories

2-category ICat:

- objects: (pseudo)functors

$$F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

- 1-morphisms: (pseudo)natural transformations



- 2-morphisms: suitable modifications

Example: Rings and Modules

A ring homomorphism $f: k \to k'$ induces a functor

$$f^*: \mathsf{Mod}_{k'} \to \mathsf{Mod}_k$$

given by $f^*(M) = M$ but with the k-action defined by

$$r.m = f(r).m$$

for $r \in k$, and

$$f^*(g) = g$$

This gives a functor $\mathsf{Mod}: \mathsf{Ring}^{\mathrm{op}} \to \mathsf{Cat}$ sending a ring to its category of modules and a morphism f to $\mathsf{Mod}_f = f^*$.

Example: semidirect product

Given an action of a group G on a group H

$$A\colon G \to \mathrm{Aut}(H)$$

you can think of it as a functor of the form

$$G \xrightarrow{A} \mathsf{Grp} \hookrightarrow \mathsf{Cat}$$

$$* \mapsto H$$

$\int A$ has:

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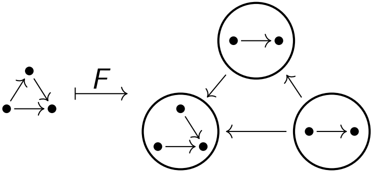
$\int A$ has:

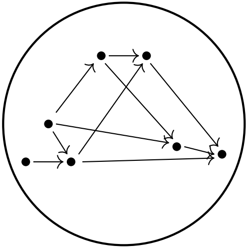
- a single object: $(*, *)$
- morphisms: (g, h) with $g \in G$, $h \in H$
- $(g, h) \circ (g', h') = (g \circ g', h \circ g.h')$

So $\int A = H \rtimes G$, the semidirect product!

2-functor $\int : \mathsf{ICat} \to \mathsf{Cat}$



$\int F =$



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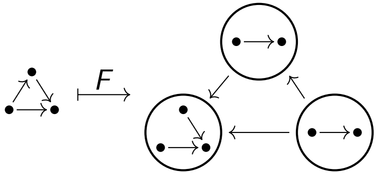

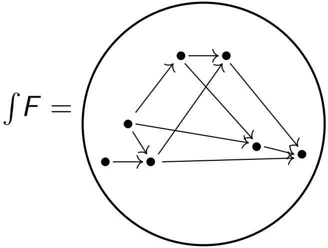

- it remembers all the objects and maps in each Fx

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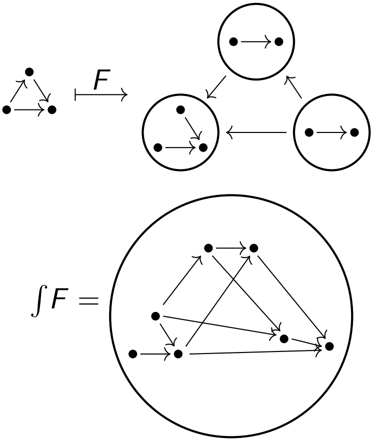

- it remembers all the objects and maps in each Fx
- it throws in extra maps representing what the functors Ff do

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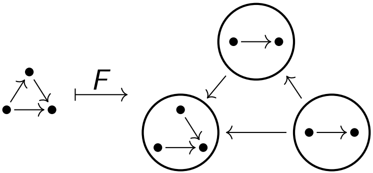
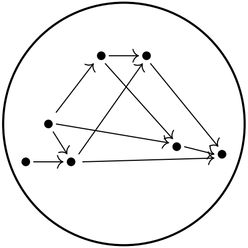


$\int F =$



- it remembers all the objects and maps in each Fx
- it throws in extra maps representing what the functors Ff do
- it forgets which maps came from categories or functors

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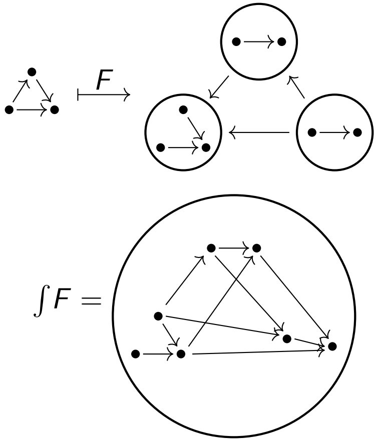


- it remembers all the objects and maps in each Fx
- it throws in extra maps representing what the functors Ff do
- it forgets which maps came from categories or functors
- We can do better!

Grothendieck Fibrations

$$\begin{array}{cccc} \mathcal{A} & & & b \\ {\scriptstyle P}\big\downarrow & & & \\ \mathcal{X} & x & \xrightarrow{\quad f \quad} & y \end{array}$$

Grothendieck Fibrations

cartesian lift

$$\begin{array}{ccccc} \mathcal{A} & \quad & a & \xrightarrow{\phi} & b \\ \downarrow{\scriptstyle P} & & & & \\ \mathcal{X} & & x & \xrightarrow{f} & y \end{array}$$

Grothendieck Fibrations

pullback

$$
\begin{array}{ccc}
\mathcal{A} & \quad & f^*(b) \xrightarrow{\;\phi\;} b \\
\Big\downarrow P & & \\
\mathcal{X} & \quad & x \xrightarrow{\;f\;} y
\end{array}
$$

Grothendieck Fibrations

reindexing functor

$$P^{-1}y \xrightarrow{f^*} P^{-1}x$$

Fibrations

2-category Fib:

- objects: fibrations $P\colon \mathcal{A} \to \mathcal{X}$
- 1-morphisms: commuting square, ϕ_t preserves cartesian morphisms

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\phi_t} & \mathcal{B} \\ {\scriptstyle P}\downarrow & & \downarrow{\scriptstyle Q} \\ \mathcal{X} & \xrightarrow[\phi_b]{} & \mathcal{Y} \end{array}$$

- 2-morphisms: suitable natural transformations

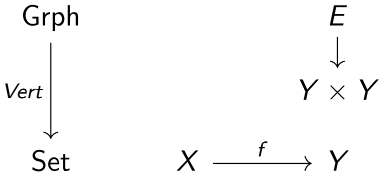
Example: Graphs

Let Grph denote the category of directed multi-graphs, each represented by a function $E \to V \times V$. Define the vertex functor

$$Vert\colon \mathrm{Grph} \to \mathrm{Set}$$

by sending a graph to its set of vertices, and a map of graphs to its vertex component. *Vert* is a fibration.

Example: Graphs



Example: Graphs

$$
\begin{array}{ccc}
\mathsf{Grph} & \qquad & \begin{array}{ccc} Q & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow \\ X \times X & \xrightarrow[f \times f]{} & Y \times Y \end{array} \\
\Big\downarrow {\scriptstyle Vert} & & \\
\mathsf{Set} & & X \xrightarrow{\quad f \quad} Y
\end{array}
$$

The Grothendieck Construction

Given

$$F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

we define $\int F$ to have

- objects (x, a), where $x \in \mathcal{X}$, $a \in F(x)$
- maps $(f, g)\colon (x, a) \to (x', a')$ where $f\colon x \to x'$ and $g\colon a \to Ff(a')$

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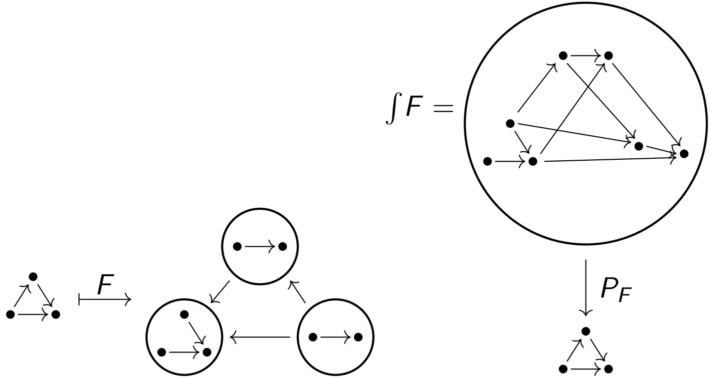
For $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$, $\int F$ is naturally fibred over $\mathcal{X}$:

$$P_F\colon \int F \to \mathcal{X}$$
$$(x, a) \mapsto x$$
$$(f, k) \mapsto f$$

2-functor $\int$: ICat $\to$ Fib



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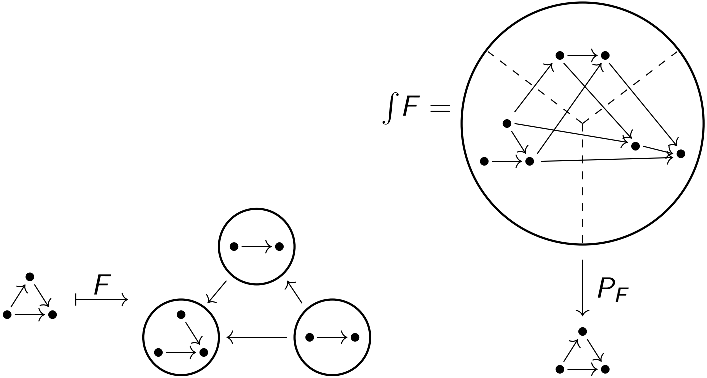


2-Equivalence

Theorem

The Grothendieck construction gives a 2-equivalence:

$$\mathsf{ICat} \cong \mathsf{Fib}$$

Example: Graphs

The fibration

$$\mathit{Vert} \colon \mathsf{Grph} \to \mathsf{Set}$$

corresponds to the indexed category

$$\mathsf{Grph}_{\_} \colon \mathsf{Set}^{\mathrm{op}} \to \mathsf{Cat}$$

where Grph_X is the category where

- the objects are graphs with fixed vertex set X
- the morphisms are map of graphs which fix the vertices.

Fixed-base Indexed Categories

2-category $\mathsf{ICat}(\mathcal{X})$:

- objects: (pseudo)functors $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$
- 1-morphisms:

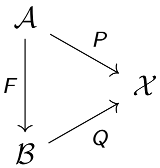
$$\mathcal{X}^{\mathrm{op}} \underset{G}{\overset{F}{\rightrightarrows}} \mathsf{Cat} \quad \Downarrow \alpha$$

- 2-morphisms: suitable modifications

Fixed-base Fibrations

2-category Fib($\mathcal{X}$):

- object
  $P\colon \mathcal{A} \to \mathcal{X}$
- 1-morphism



- 2-morphisms: suitable natural transformations

2-Equivalence for Fixed-base

Theorem

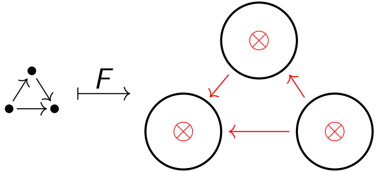
For a category $\mathcal{X}$, the Grothendieck construction gives a 2-equivalence:

$$\mathsf{ICat}(\mathcal{X}) \cong \mathsf{Fib}(\mathcal{X})$$

Fibre-wise Monoidal Indexed Categories

Definition 1

A **fibre-wise monoidal indexed category** is a pseudofunctor $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{MonCat}_s$. Let $f\mathsf{MonICat}(\mathcal{X})$ denote the 2-category of fibre-wise monoidal indexed categories



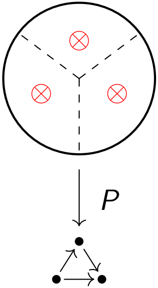
Fibre-wise Monoidal Fibrations

Definition 2

A **(fibre-wise) monoidal fibration** is

- fibration $P\colon \mathcal{A} \to \mathcal{X}$
- the fibres $\mathcal{A}_x$ are monoidal
- the reindexing functors are monoidal

Let fMonFib$(\mathcal{X})$ denote the 2-category of fibre-wise monoidal fibrations.



Fibre-wise Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to a 2-equivalence:

$$f\mathsf{MonFib}(\mathcal{X}) \simeq f\mathsf{MonICat}(\mathcal{X})$$

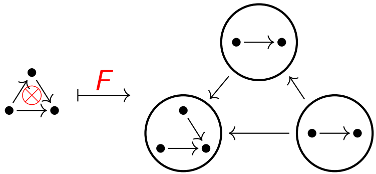
Global Monoidal Indexed Categories

Definition 3

A **(global) monoidal indexed category** is

- an indexed category $F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$
- $\mathcal{X}$ is monoidal
- F is lax monoidal $(F, \phi)\colon (\mathcal{X}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$

Let $g\mathsf{MonICat}$ denote the 2-category of global monoidal indexed categories.



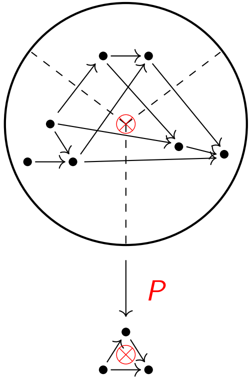
Global Monoidal Fibrations

Definition 4

A **(global) monoidal fibration** is a fibration $P\colon \mathcal{A} \to \mathcal{X}$

- $\mathcal{A}$ and $\mathcal{X}$ are monoidal
- P is a strict monoidal functor
- $\otimes_{\mathcal{A}}$ preserves cartesian liftings.

Let $g\mathrm{MonFib}(\mathcal{X})$ denote the 2-category of global monoidal fibrations.



Global Monoidal Grothendieck Construction

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$g\mathsf{MonFib}(\mathcal{X}) \simeq g\mathsf{MonICat}(\mathcal{X})$$

Monoidal structure on the total category

Given a lax monoidal functor

$$(F, \phi)\colon (\mathcal{X}^{\mathrm{op}}, \otimes) \to (\mathsf{Cat}, \times)$$

$$\phi\colon Fx \times Fy \to F(x \otimes y)$$

$$(x, a) \otimes (y, b) =$$

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$$(x, a) \otimes (y, b) = (x \otimes y, \phi_{x,y}(a, b))$$

Cartesian Case

Theorem (Vasilakopoulou, M)

If $\mathcal{X}$ is a cartesian monoidal category, then

$$\begin{array}{ccc} g\mathsf{MonFib}(\mathcal{X}) & \xrightarrow{\simeq} & g\mathsf{MonICat}(\mathcal{X}) \\ \wr\!\!\downarrow & & \downarrow\!\wr \\ f\mathsf{MonFib}(\mathcal{X}) & \xrightarrow{\simeq} & f\mathsf{MonICat}(\mathcal{X}) \end{array}$$

Dually, if $\mathcal{X}$ is cocartesian, then

$$\begin{array}{ccc} g\mathsf{MonOpFib}(\mathcal{X}) & \xrightarrow{\simeq} & g\mathsf{MonOpICat}(\mathcal{X}) \\ \wr\!\!\downarrow & & \downarrow\!\wr \\ f\mathsf{MonOpFib}(\mathcal{X}) & \xrightarrow{\simeq} & f\mathsf{MonOpICat}(\mathcal{X}) \end{array}$$

$g\mathsf{MonICat}(\mathcal{X}) \to f\mathsf{MonICat}(\mathcal{X})$

Given $(F, \phi)\colon (X, \times)^{\mathrm{op}} \to (\mathsf{Cat}, \times)$
define $\otimes_x\colon Fx \times Fx \to Fx$ by

$$\begin{array}{ccc} Fx \times Fx & \xrightarrow{\quad \otimes_x \quad} & Fx \\ & {\scriptstyle \phi_{x,x}} \searrow \qquad \nearrow {\scriptstyle F\Delta_x} & \\ & F(x \times x) & \end{array}$$

$f\mathsf{MonICat}(\mathcal{X}) \to g\mathsf{MonICat}(\mathcal{X})$

Given $F\colon X^{\mathrm{op}} \to \mathsf{MonCat}$
define $\phi_{x,y}\colon Fx \times Fy \to F(x \times y)$ by

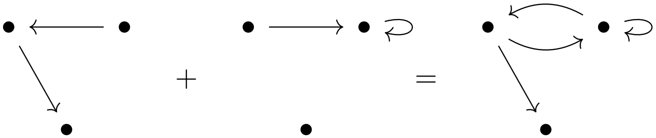
$$Fx \times Fy \xrightarrow{\phi_{x,y}} F(x \times y)$$

$$Fx \times Fy \xrightarrow{F\pi_x \times F\pi_y} F(x \times y) \times F(x \times y) \xrightarrow{\otimes_{x \times y}} F(x \times y)$$

Example: Graphs

(cocartesian) monoidal structure on Grph(X) given by

$$(f\colon E \to X^2) +_X (f'\colon E' \to X^2) = (\langle f, f' \rangle \colon E + E' \to X^2)$$



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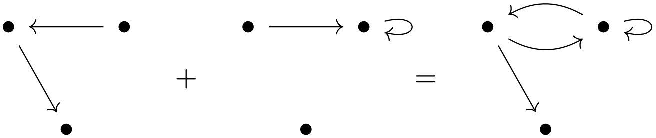
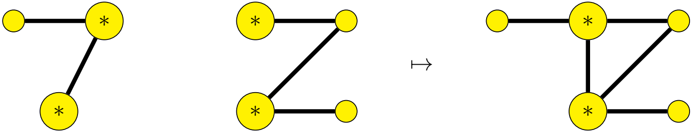


Then we can define a lax monoidal structure on $\mathsf{Grph}(-)\colon \mathsf{Set} \to \mathsf{Cat}$ by

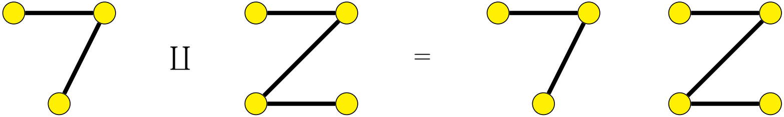
$$\begin{array}{ccc} \mathsf{Grph}(X) \times \mathsf{Grph}(Y) & \xrightarrow{\quad + \quad} & \mathsf{Grph}(X+Y) \\ & {}_{i_* \times j_*}\searrow \qquad \nearrow_{+_{X+Y}} & \\ & \mathsf{Grph}(X+Y) \times \mathsf{Grph}(X+Y) & \end{array}$$

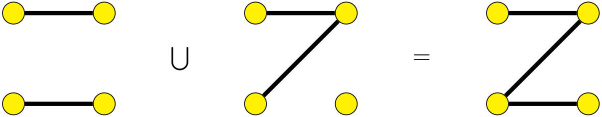
Constructing network operads

Graphs can be combined to create bigger graphs by identifying some of the vertices

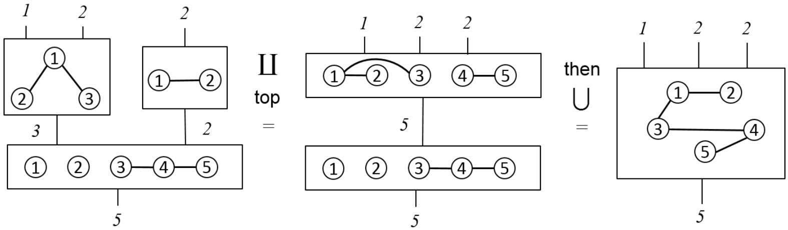


We choose to examine these as combinations of a few simpler operations





We want to construct an operad that captures these operations



Types of networks as functors

Simple graphs, as a symmetric lax monoidal functor:

$$(SG, \sqcup) \colon (\mathsf{FinBij}, +) \to (\mathsf{Mon}, \times)$$

- $\mathrm{SG}(n)$ simple graphs with vertex set n

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- monoid operation $\cup\colon \mathrm{SG}(n) \times \mathrm{SG}(n) \to \mathrm{SG}(n)$ given by "overlaying" two graphs

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- monoid operation $\cup\colon \mathrm{SG}(n) \times \mathrm{SG}(n) \to \mathrm{SG}(n)$ given by "overlaying" two graphs
- lax structure $\sqcup\colon \mathrm{SG}(n) \times \mathrm{SG}(m) \to \mathrm{SG}(n+m)$

Types of networks as functors

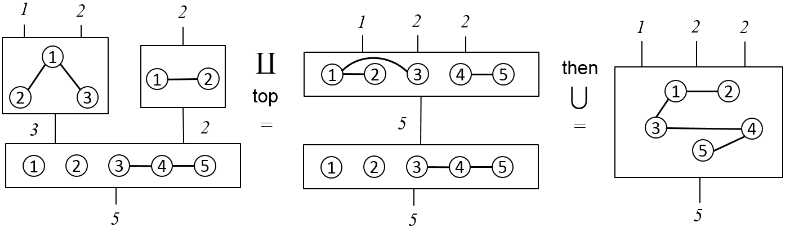
A **network model** is a symmetric lax monoidal functor

$$(F, \phi)\colon (\mathsf{FinBij}, +) \to (\mathsf{Mon}, \times) \hookrightarrow (\mathsf{Cat}, \times)$$

Examples:

- Multigraphs
- Directed Graphs
- Partitions
- Graphs with colored vertices
- Petri Nets
- Graphs with edges weighted by a monoid

$$\mathsf{NetMod} \xrightarrow{\int} \mathsf{SymMonCat} \xrightarrow{op} \mathsf{Operad}$$



The critical difference between ICat and ICat($\mathcal{X}$)

products in ICat:

$$F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$
$$G\colon \mathcal{Y}^{\mathrm{op}} \to \mathsf{Cat}$$

$$\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}} \xrightarrow{F\times G} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat}$$

The critical difference between ICat and $\mathsf{ICat}(\mathcal{X})$

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$$F\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$
$$G\colon \mathcal{Y}^{\mathrm{op}} \to \mathsf{Cat}$$

$$\mathcal{X}^{\mathrm{op}} \times \mathcal{Y}^{\mathrm{op}} \xrightarrow{F\times G} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat}$$

products in $\mathsf{ICat}(\mathcal{X})$:

$$F, G\colon \mathcal{X}^{\mathrm{op}} \to \mathsf{Cat}$$

$$\mathcal{X}^{\mathrm{op}} \xrightarrow{\Delta} \mathcal{X}^{\mathrm{op}} \times \mathcal{X}^{\mathrm{op}} \xrightarrow{F\times G} \mathsf{Cat} \times \mathsf{Cat} \xrightarrow{\times} \mathsf{Cat}$$

The proof

We wrote the definitions so that

- $f\mathsf{MonICat}(\mathcal{X}) \cong \mathsf{PsMon}(\mathsf{ICat}(\mathcal{X}))$.
- $f\mathsf{MonFib}(\mathcal{X}) \cong \mathsf{PsMon}(\mathsf{Fib}(\mathcal{X}))$.
- $g\mathsf{MonICat} \cong \mathsf{PsMon}(\mathsf{ICat})$.
- $g\mathsf{MonFib} \cong \mathsf{PsMon}(\mathsf{Fib})$.

If $A \cong B$, then $\mathsf{PsMon}(A, \times) \cong \mathsf{PsMon}(B, \times)$.

Braided and Symmetric

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to equivalences:

$$g\mathsf{BrMonFib}(\mathcal{X}) \simeq g\mathsf{BrMonICat}(\mathcal{X})$$

$$f\mathsf{BrMonFib}(\mathcal{X}) \simeq f\mathsf{BrMonICat}(\mathcal{X})$$

Theorem (Vasilakopoulou, M)

The Grothendieck construction lifts to an equivalence:

$$g\mathsf{SymMonFib}(\mathcal{X}) \simeq g\mathsf{SymMonICat}(\mathcal{X})$$

$$f\mathsf{SymMonFib}(\mathcal{X}) \simeq f\mathsf{SymMonICat}(\mathcal{X})$$

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